Advanced Algorithm

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May 16, 2019

k-Center Problem

- Ref: Approximation Algorithm - Chapter 5
- undirected graph $G = (V, E)$ with non-negative edge cost, find k nodes S to minimize $\max_v dist(v, S)$. Here $dist(v, S) = \min_{u \in S} w(u, v)$.
- General case: cannot be approximated within any computable factor $\alpha(n)$ (Homework)
- Metric k-center problem
  - Algorithm: based on dominating set
  - Analysis: 2-approximation ratio, tight
- Hardness result: $(2 - \epsilon)$ - inapproximation, if $P \neq NP$.

Homework

- Approximation Algorithm - Problem 5.1, page 52 (k-Center)

Lecture 9: Introduction to LP-Duality

Linear programming

- Ref: Approximation Algorithm - Chapter 12
- Linear program (prime program)

$$\begin{aligned} min \quad & c^T x \\ s.t \quad & Ax \geq b \\ & x_i \geq 0 \end{aligned}$$

- Simplex Algorithm
  - Practical, but exponential time in the worst-case
- Ellipsoid Algorithm
  - First polynomial time algorithm, but slow in practice
- Karmarkars Algorithm (interior point)
  - Polynomial time algorithm and competitive in practice
- Software: LINDO, CPLEX, Solver (in Excel)

Weighted Vertex Cover

- (weighted) Vertex Cover

$$\begin{array}{lll} min & \sum_v w(v)x_v & \\ s.t & x_i + x_j \geq 1 & \forall (i,j) \in E \\ & x_i \in \{0,1\} & \end{array}$$

- integer programming $\rightarrow$ linear programming
- $x_i \in \{0,1\} \rightarrow 0 \leq x_i \leq 1$
- Design approximation algorithm based on the optimal solution of LP

LP-based approximation

- State the problem as an integer programming
- Linear relaxation: integer program $\to$ linear program
  - Maximize problem: OPT $\uparrow$; Minimize problem: OPT $\downarrow$
- Method 1: linear-rounding
- Method 2: primal-dual schema
- Analysis: integrality gap
- For minimize problem, $ALGO \leq \alpha OPT_{LP} \leq \alpha OPT_{IP}$

Dual Program

- Linear program (prime program)

$$\begin{aligned} min \quad & c^T x \\ s.t \quad & Ax \geq b \\ & x_i \geq 0 \end{aligned}$$

- Dual program

$$\begin{aligned} max \quad & b^T y \\ s.t \quad & A^T y \leq c \\ & y_j \geq 0 \end{aligned}$$

- LP-duality theorem

LP duality

Theorem (Weak duality theorem)

If $x = (x_1, \cdots, x_n)$ *and* $y = (y_1, \cdots, y_n)$ *are feasible solutions for the primal and dual program, respectively, then,* $\sum_j c_j x_j \geq \sum_i b_i y_i$.

Theorem (Complementary slackness conditions)

Let x *and* y *be primal and dual feasible solutions, respectively.*
Then, x *and* y *are both optimal iff all of the following conditions are satisfied:*
Primal complementary slackness conditions
For each $1 \leq j \leq n$: *either* $x_j = 0$ *or* $\sum_i a_{ij} y_i = c_j$;
Dual complementary slackness conditions
For each $1 \leq i \leq m$: *either* $y_i = 0$ *or* $\sum_j a_{ij} x_j = b_i$.

Weighted Set Cover problem - primal dual schema

- Ref: Approximation Algorithm - Chapter 13

$$\begin{array}{ll} min & \sum_{S\in\mathfrak{S}} c(S)x_S \\ s.t & \sum_{S:e\in S} x_S \geq 1 \quad \forall e\in U \\ & x_S \in \{0,1\} \end{array}$$

- LP-relaxation

$$\begin{array}{ll} min & \sum_{S\in\mathfrak{S}} c(S)x_S \\ s.t & \sum_{S:e\in S} x_S \geq 1 \quad \forall e\in U \\ & x_S \geq 0 \end{array}$$

- Dual-fitting based analysis for the greedy algorithm
- $\log n$-approximation
- integrality gap